Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER

Journal of Sound and Vibration 282 (2005) 381–399

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Solution theorems for the standard eigenvalue problem of structures with uncertain-but-bounded parameters

Zhiping Qiu*, Xiaojun Wang

Institute of Solid Mechanics, Beijing University of Aeronautics and Astronautics, Beijing 100083, P.R. China

Received 30 June 2003; received in revised form 26 January 2004; accepted 23 February 2004
Available online 28 September 2004

Abstract

Generalized eigenvalue problems from the modal analysis are often converted to the standard eigenvalue problems. In this paper, it evaluates the upper and lower bounds on the eigenvalues of the standard eigenvalue problem of structures subject to severely deficient information about the structural parameters. Here, we focus on non-probabilistic interval analysis models of uncertainty, which are adapted to the case of severe lack of information on uncertainty. Non-probabilistic, interval analysis method in which uncertainties are defined by interval numbers appears as an alternative to the classical probabilistic models. For the standard eigenvalue problem of structures with uncertain-but-bounded parameters, the vertex solution theorem, the positive semi-definite solution theorem and the parameter decomposition solution theorem for the standard eigenvalue problem are presented, and compared with Deif's solution theorem in numerical examples. It is shown that, for the upper and lower bounds on the eigenvalues of the standard eigenvalue problem with uncertain-but-bounded parameters, the presented vertex solution theorem is unconditional, and the positive semi-definite solution theorem and the parameter decomposition solution theorem have less limitary conditions compared with Deif's solution theorem. The effectiveness of the vertex solution theorem, the positive semi-definite solution theorem and the parameter decomposition solution theorem are illustrated by numerical examples

*Corresponding author.
E-mail address: zpqiu@buaa.edu.cn (Z. Qiu).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2004.02.024

1. Introduction

In practical engineering and scientific researches, uncertainty in the quantities of interest is the rule rather than exception. In particular, all engineering analysis and design problems involve uncertainty to varying degrees. Depending on the nature of uncertainty, probabilistic and statistical models, fuzzy mathematics, and non-probabilistic convex models and interval analysis methods are all used to deal with uncertain problems. Probabilistic and statistical models are the most widely used methods of dealing with uncertain problems in most engineering and scientific researches. On the one hand, when the probability density function is known for the uncertain variables under investigation, either from a fundamental understanding of the sources of uncertainty or from extensive statistical measurements, this technique is very powerful and gives well-defined results for specified levels of reliability. On the other hand, in many practical problems in engineering and science, the sources of the uncertainty are too complex to allow analytical determination of the probability density function, while in the same instance it is impractical to obtain enough data to determine the probability density functions empirically. In these situations, the probability density functions are often simply assumed by analogy to similar problems for which probability densities are known. This leads to questionable reliability estimates, particularly for high levels of reliability associated with the tails of the density functions, which are most questionable for assumed probability density functions. Despite the success of probabilistic and statistical models used for uncertain problems in most engineering fields, one may recognize that uncertainty is not randomness; uncertain problem can be modeled on the basis of alternative, non-probabilistic conceptual frameworks. One motivation for using non-probabilistic convex models and interval analysis methods rather than probabilistic and statistical models for uncertain problems is the general dearth of information in characterizing the uncertain variable. Non-probabilistic convex models and interval analysis methods are less information-intensive than probabilistic and statistical models, since no density information is required. In non-probabilistic convex models and interval analysis methods, the uncertain quantities are not modeled as random variables or stochastic processes, but are considered instead to be unknown except that they belong to given sets in an appropriate vector space. In this case, all information about the structural system response is provided by the set of responses consistent with the constraints on the uncertain quantities.

Based on interval mathematics or interval analysis, interval analysis models are developed by many researchers in order to solve uncertain problems. In this kind of models, the uncertain variables can be quantified by an interval number or vector, the calculated result is also interval number or vector. Interval mathematics is discussed in a number of books [1,2]. In recent work of interval analysis models, bounds on the magnitude of uncertain variables are only required, not necessarily knowing the probabilistic distribution densities, following the general methodologies developed in the monographs. It was assumed that the uncertain variables fall into the multidimensional box, instead of conventional optimization studies, where the minimum possible response is sought, here an uncertainty modeling is developed as an anti-optimization problem of finding the least-favorable response and the most favorable response under the constrains within the set-theoretical description. Convex models and interval analysis methods have been used for dealing with uncertain phenomena in a wide range of engineering applications.

The interval eigenvalue problem emerged in recent years as scientists and engineers started to realize its wide applicability. Here, we make a brief review. Rohn [3] studied the standard interval eigenvalue problem of the symmetric interval matrix and derived formulas of interval eigenvalues when the deviation radius matrix had rank one. Hallot and Bartlett [4] found that the spectrum of eigenvalue of an interval matrix family depends on the spectrum of its extremes sets. Hudak [5] investigated ways of finding a constant matrix under the certain condition. Based on the invariance properties of the characteristic vector's entries, Deif [6] presented a method of computing interval eigenvalues for the standard interval eigenvalue problem. Qiu et al. [7] have extended Deif's method to the generalized interval eigenvalue problem. Because there exists no efficient criterion for judging invariance properties of the signs of the components of the eigenvector under the interval operations before computing interval eigenvalues, applications of Deif's approach appear to be restricted. In order to eliminate the limitation of Deif's method about the signs of components of eigenvectors, by means of assumption of positive semi-definiteness of the deviation radius interval matrix pair of the interval matrix pair, Qiu et al. [8] developed a method for computing interval eigenvalues. Under small deviation radii of the interval matrix, Qiu et al. [9] also presented an interval perturbation method for the interval eigenvalue problem.

In this paper, considering the characteristics of the engineering structure, the vertex solution theorem, the positive semi-definite solution theorem and the parameter decomposition solution theorem are presented, and compared with Deif's solution theorem by two numerical examples.

2. Problem formulation

The algebraic eigenvalue problem in finite element analysis is usually determined from the conservative part of the structural system equations of motion whose eigenvalue equation is given in the following matrix form:

$$Ku = \lambda Mu, \tag{1}$$

where $K = (k_{ij})$ is the $n \times n$-dimensional symmetric stiffness matrix and $M = (m_{ij})$ is the $n \times n$-dimensional symmetric positive definite mass matrix, λ and u are the eigenvalue and the associated eigenvector, respectively.

The eigenvalues given by Eq. (1) are usually assumed to be constants for identical structural systems. However, experience and experiments have shown that these values vary uncertainly because in reality the physical and geometric properties of the elements in K and M can neither be measured exactly nor manufactured exactly. In this paper, we assume that the uncertainties in K and M are bounded, and the uncertain-but-bounded matrices K and M can be written as the following matrix inequality form:

$$\underline{K} \leq K \leq \overline{K}, \quad \underline{M} \leq M \leq \overline{M} \tag{2}$$

in which $\overline{K} = (\overline{k}_{ij})$ and $\underline{K} = (\underline{k}_{ij})$, respectively, are the upper bound matrix and the lower bound matrix of the uncertain stiffness matrix K, and $\overline{M} = (\overline{m}_{ij})$ and $\underline{M} = (\underline{m}_{ij})$, respectively, are the upper bound matrix and the lower bound matrix of the uncertain mass matrix M.

Alternatively, the algebraic eigenvalue problem can be stated by transforming Eq. (1) into the standard eigenvalue form

$$Au = \lambda u, \tag{3}$$

where $A = M^{-1}K = (a_{ij})$ is the $n \times n$-dimensional structural system dynamic matrix which is symmetric positive definite and has the uncertain-but-bounded matrix inequality form

$$\underline{A} \le A \le \overline{A} \tag{4a}$$

or the element form

$$\underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, \ \ i,j = 1, 2, \ldots, n \tag{4b}$$

in which $\overline{A} = (\overline{a}_{ij})$ and $\underline{A} = (\underline{a}_{ij})$, respectively, are the upper bound matrix and the lower bound matrix of the uncertain-but-bounded structural system dynamic matrix A.

It is very difficult to solve the standard eigenvalue problem (3) under the condition of the matrix inequality constraint conditions (4).

We will confine ourselves to the case in which the eigenvalues of A are all distinct, so that the eigenvectors of A are all independent.

By virtue of the interval matrix notation [1,2], the matrix inequality constraint conditions can be written as

$$A \in A^I = [\underline{A}, \overline{A}] = (a_{ij}^I), \quad a_{ij}^I = [\underline{a}_{ij}, \overline{a}_{ij}], \quad i,j = 1, 2, \ldots, n, \tag{5}$$

where A_I is the $n \times n$-dimensional real symmetric interval matrix.

Thus, the standard eigenvalue problem (3) under the constraint conditions of the matrix inequality (4a) can be expressed in the following compact and simple form

$$A^I u = \lambda u. \tag{6}$$

Eq. (6) is called the standard interval eigenvalue problem [6].

For the sake of convenience, the standard interval eigenvalue problem may be stated as follows: given the upper bound $\overline{A} = (\overline{a}_{ij})$ and lower bound $\underline{A} = (\underline{a}_{ij})$ of the uncertain but bounded matrix $A = (a_{ij})$, to find all possible eigenvalues satisfying $Au = \lambda u$ subject to $A \in A^I = [\underline{A}, \overline{A}]$. Obviously, the infinite number of eigenvalues constitutes a region in real number field R, which is denoted by Γ, i.e.

$$\Gamma = \left\{ \lambda : \lambda \in R, Au = \lambda u, A \in A^I \right\}. \tag{7}$$

In general, the computation of region (7) is extremely difficult, because Γ has a very complicated region and needn't be convex. Taking this into account, one has to determine a closed convex hull $[\underline{\lambda}_i, \overline{\lambda}_i]$ for each eigenvalue which is the narrowest one enclosing all possible values satisfying the standard interval eigenvalue equation (6). Thus, the kind of the interval estimation can be written in the following form:

$$\lambda_i \in \lambda_i^I = [\underline{\lambda}_i, \overline{\lambda}_i], \quad i = 1, 2, \ldots, n \tag{8}$$

where

$$\overline{\lambda}_i = \max_{A \in A^I} \left\{ \lambda_i(A) \right\}, \quad \underline{\lambda}_i = \min_{A \in A^I} \left\{ \lambda_i(A) \right\}, \quad i = 1, 2, \ldots, n \tag{9}$$

in which

$$\lambda_i = \lambda_i(A) = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{u^{\mathrm{T}} A u}{u^{\mathrm{T}} u} \right\}, \tag{10}$$

where $\Phi_i \subset R^n$ is an arbitrary i-dimensional sub-space.

Obviously, the maximum and minimum eigenvalue problems in Eqs. (9) are all global optimization problems.

3. Deif's solution theorem

Based on the interval matrix, we may define the midpoint or nominal value or mean value of the interval matrix $A^I = [\underline{A}, \overline{A}]$

$$A^c = \frac{(\overline{A} + \underline{A})}{2} = (a_{ij}^c), \quad a_{ij}^c = \frac{(\overline{a}_{ij} + \underline{a}_{ij})}{2}, \quad i,j = 1,2,\ldots,n \tag{11}$$

and the uncertain radius or deviation amplitude or uncertainty of the interval matrix $A^I = [\underline{A}, \overline{A}]$

$$\Delta A = \frac{(\overline{A} - \underline{A})}{2} = (\Delta a_{ij}), \quad \Delta a_{ij} = \frac{(\overline{a}_{ij} - \underline{a}_{ij})}{2}, \quad i,j = 1,2,\ldots,n. \tag{12}$$

For the standard interval eigenvalue problem of the real symmetric interval matrix, Professor Deif once gave a solution theorem [6]. This theorem can be stated as follows:

Deif'ssolution theorem. If $A^I = [\underline{A}, \overline{A}] = [A^c - \Delta A, A^c + \Delta A]$ is a real symmetric interval matrix and the signal matrix

$$S^i = \mathrm{diag}(\mathrm{sgn}(u_{1i}),\ \mathrm{sgn}(u_{2i}), \ldots, \mathrm{sgn}(u_{ni})), \quad i = 1,2,\ldots,n \tag{13}$$

is constant over A^I, where $u_i = (u_{ki}), i = 1,2,\ldots,n$, are eigenvectors of $A \in A^I$ and the eigenvectors u_i have been normalized so as to satisfy

$$u_i^{\mathrm{T}} u_j = \delta_{ij}, \quad i,j = 1,2,\ldots,n. \tag{14}$$

Then the eigenvalues $\lambda_i, i = 1,2,\ldots,n$, of $A \in A^I$ range over the intervals

$$\lambda_i^I = [\underline{\lambda}_i, \overline{\lambda}_i], \quad i = 1,2,\ldots,n, \tag{15}$$

where the lower bound eigenvalues $\underline{\lambda}_i, i = 1,2,\ldots,n$, satisfy

$$(A^c - S^i \Delta A S^i)\underline{u}_i = \underline{\lambda}_i \underline{u}_i, \quad i = 1,2,\ldots,n \tag{16}$$

in which $\underline{u}_i$ is the eigenvector associated with the eigenvalue $\underline{\lambda}_i$, and the upper bound eigenvalues $\overline{\lambda}_i, i = 1,2,\ldots,n$, satisfy

$$(A^c + S^i \Delta A S^i)\overline{u}_i = \overline{\lambda}_i \overline{u}_i, \quad i = 1,2,\ldots,n \tag{17}$$

in which is the eigenvector associated with the eigenvalue $\overline{\lambda}_i$.

Remarks. (a) It can be seen from Eq. (11) that $u_{ki} \neq 0, k, i = 1,2,\ldots,n$, must hold.

(b) the condition (14) is added by us, because the eigenvectors can be determined $u_i, i = 1,2,\ldots,n$ only in such a manner, and Eq. (13) may be significant.

(c) The interval eigenvalues $\lambda_i^I = [\underline{\lambda}_i, \overline{\lambda}_i], i = 1, 2, \ldots, n$, of the interval matrix $A^I = [\underline{A}, \overline{A}] = [A^c - \Delta A, A^c + \Delta A]$ can be determined by solving $2n$ standard eigenvalue problems of the matrices $(A^c - S^i \Delta A S^i), (A^c + S^i \Delta A S^i), i = 1, 2, \ldots, n$. Clearly, the computational efforts of Deif's solution theorem are quite large.

(d) It is quite difficult to determine the invariance properties of the eigenvectors' components in the interval matrix.

4. The vertex solution theorem

Obviously, the region expressed by the interval matrix $A^I = [\underline{A}, \overline{A}] = (a_{ij}^I)$ is a convex set.

From the interval matrix, the boundary matrix or extreme point matrix or vertex matrix of an interval matrix, $A^I = [\underline{A}, \overline{A}]$ is defined by

$$\hat{A}_s = \left\{ \hat{A}_s = (\hat{a}_{ij}^s) \in A^I : \hat{a}_{ij}^s = \overline{a}_{ij} \text{ or } \hat{a}_{ij}^s = \underline{a}_{ij}, \overset{\frown s}{a}_{ij} = \overset{\frown s}{a}_{ij}, i, j = 1, 2, \ldots, n \right\}, \quad s = 1, 2, \ldots, 2^{n \times n}. \tag{18}$$

Under the matrix inequality constraint condition (4a), let us consider the minimax Rayleigh quotient of the $n \times n$-dimensional real symmetric matrix $A = (a_{ij})$

$$\lambda_i = \lambda_i(A) = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{u^T A u}{u^T u} \right\}, \quad i = 1, 2, \ldots, n. \tag{19}$$

Thus, $\lambda_i, i = 1, 2, \ldots, n$, are all real-valued functions defined by $A = (a_{ij})$, since the matrix $A = (a_{ij})$ is a real symmetric.

According to the definition of the quadratic form and the inner product of two vectors, the minimax Rayleigh quotient may also be written in the following element form

$$\lambda_i = \lambda_i(A) = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{\sum_{k,\ l=1}^{n} a_{kl} u_k u_l}{\sum_{j=1}^{n} u_j^2} \right\}, \quad i = 1, 2, \ldots, n \tag{20}$$

subject to the element inequality constraint condition (4b).

Expression (20) and inequalities (4b) can be simply written as the extremum value problem

$$\lambda_{iext}(A) = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \underset{\substack{a_{kl} \in a_{kl}^I \\ k,l = 1,2,\ldots,n}}{\text{extremum}} \left\{ \frac{\sum_{k,\ l=1}^{n} a_{kl} u_k u_l}{\sum_{j=1}^{n} u_j^2} \right\} \right\}, \quad i = 1, 2, \ldots, n \tag{21}$$

From the above relation, we can see that the eigenvalues $\lambda_i, i = 1, 2, \ldots, n$, are all linear functions of the elements $a_{ij}, i, j = 1, 2, \ldots, n$.

According to the optimum theory, the extreme value problem

$$R = \underset{\substack{a_{kl} \in a_{kl}^I \\ k,l=1,2,\ldots,n}}{\text{extremum}} \left\{ \sum_{k,\ l=1}^{n} a_{kl} u_k u_l \bigg/ \sum_{j=1}^{n} u_j^2 \right\}$$

is essentially the extreme value problem

$$T = \underset{\substack{a_{kl} \in a_{kl}^I \\ k,l=1,2,\ldots,n}}{\text{extremum}} \left\{ \sum_{k,\ l=1}^{n} a_{kl} u_k u_l \right\},$$

i.e.

$$R = \underset{\substack{a_{kl} \in a_{kl}^I \\ k,l=1,2,\ldots,n}}{\text{extremum}} \left\{ \frac{\sum_{k,\ l=1}^{n} a_{kl} u_k u_l}{\sum_{j=1}^{n} u_j^2} \right\} = \frac{T}{\sum_{j=1}^{n} u_j^2} = \frac{\underset{\substack{a_{kl} \in a_{kl}^I \\ k,l=1,2,\ldots,n}}{\text{extremum}} \left\{ \sum_{k,\ l=1}^{n} a_{kl} u_k u_l \right\}}{\sum_{j=1}^{n} u_j^2}. \tag{22}$$

For the extreme value problem

$$T = \underset{\substack{a_{kl} \in a_{kl}^I \\ k,l=1,2,\ldots,n}}{\text{extremum}} \left\{ \sum_{k,\ l=1}^{n} a_{kl} u_k u_l \right\},$$

know that the quantity T is a linear function of the elements $a_{kl}, k, l = 1, 2, \ldots, n$. Based on the extreme theorem in convex analysis, since the quantity T is a convex (or concave) function of the elements $a_{kl}, k, l = 1, 2, \ldots, n$, and the interval sets $a_{kl}^I = [\underline{a}_{kl}, \overline{a}_{kl}], k, l = 1, 2, \ldots, n$, are all convex, the extreme values of T will be reached on the boundary matrix or vertex matrix of the interval stiffness matrix $A^I = [\underline{A}, \overline{A}] = (a_{ij}^I)$, i.e.

$$T = \underset{\substack{a_{kl} \in a_{kl}^I \\ k,l=1,2,\ldots,n}}{\text{extremum}} \left\{ \sum_{k,\ l=1}^{n} a_{kl} u_k u_l \right\} = \sum_{k,\ l=1}^{n} \hat{a}_{kl}^s u_k u_l = u^{\mathrm{T}} \hat{A}_s u.$$

Thus, from the expression (22), we obtain

$$R = \underset{\substack{a_{kl} \in a_{kl}^I \\ k,l=1,2,\ldots,n}}{\text{extremum}} \left\{ \frac{\sum_{k,\ l=1}^{n} a_{kl} u_k u_l}{\sum_{j=1}^{n} u_j^2} \right\} = \frac{u^{\mathrm{T}} \hat{A}_s u}{u^{\mathrm{T}} u}. \tag{23}$$

Substitution of Eq. (23) into Eq. (21) yielding

$$\lambda_{is} = \lambda_{i\text{ext}}(\hat{A}_s) = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{u^{\mathrm{T}} \hat{A}_s u}{u^{\mathrm{T}} u} \right\}, \quad s = 1, 2, 3, \ldots, 2^{n \times n}, \quad i = 1, 2, \ldots, n. \tag{24}$$

According to the optimization theory in convex analysis, since the eigenvalues $\lambda_i, i = 1, 2, \ldots, n$, are all convex (or concave) functions of the elements $a_{ij}, i, j = 1, 2, \ldots, n$, and the set $a_{ij}^I =$

$[\underline{a}_{ij}, \overline{a}_{ij}], i, j = 1, 2, \ldots, n$, are convex, the maximum and minimum values of $\lambda_i, i = 1, 2, \ldots, n$, occur on the boundary matrix or extreme point matrix or vertex matrix of the interval matrix $A^I = [\underline{A}, \overline{A}] = (a^I_{ij})$, i.e.

$$\overline{\lambda}_i = \lambda_{i\max} = \max_{1\leq s\leq 2^{n\times n}}\{\lambda_i(\hat{A}_s)\}, \quad \underline{\lambda}_i = \lambda_{i\min} = \min_{1\leq s\leq 2^{n\times n}}\{\lambda_i(\hat{A}_s)\}, \quad i = 1, 2, \cdots, n, \tag{25}$$

where

$$\lambda_{is} = \lambda_i(\hat{A}_s) = \min_{\Phi_i\subset R^n}\max_{\substack{u\in\Phi_i\\ u\neq 0}}\left\{\frac{u^{\mathrm{T}}\hat{A}_s u}{u^{\mathrm{T}}u}\right\} = \min_{\Phi_i\subset R^n}\max_{\substack{u\in\Phi_i\\ u\neq 0}}\left\{\frac{\sum\limits_{k,\ l=1}^{n}\hat{a}^s_{kl}u_k u}{\sum\limits_{j=1}^{n}u_j^2}\right\}$$

$$s = 1, 2, \ldots, 2^{n\times n}, \quad i = 1, 2, \ldots, n, \tag{26}$$

The stationary condition of Rayleigh's quotient is equivalent to the algebraic eigenvalue problem [10,11]. Thus, the eigenvalue problem corresponding to Eq. (26) reads

$$\hat{A}_s u_{is} = \lambda_{is}u_{is}, \quad s = 1, 2, \ldots, 2^{n\times n}, \quad i = 1, 2, \ldots, n, \tag{27}$$

where $\hat{A}_s = (\hat{a}^s_{ij})$, and u_{is} is the eigenvector associated with the ith eigenvalue λ_{is}.

Thus, we arrive at the following:

The vertex solution theorem. *If an interval matrix $A^I = [\underline{A}, \overline{A}] = (a^I_{ij})$ is real symmetric, and its vertex matrix is expressed as $\hat{A}_s = (\hat{a}^s_{ij})$, where $\hat{a}^s_{ij} = \underline{a}_{ij}$ or $a^s_{ij} = \overline{a}_{ij}$, $\underset{\smile}{a}^s_{ij} = \overset{\frown}{a}^s_{ij}, i, j = 1, 2, \ldots, n$; $s = 1, 2, \ldots, 2^{n\times n}$. Then the interval eigenvalues λ_i, $i = 1, 2, \ldots, n$, of the real symmetric interval matrix can be determined as follows*:

$$\lambda^I_i = [\underline{\lambda}_i, \overline{\lambda}_i], \quad i = 1, 2, \ldots, n, \tag{28}$$

where the upper bound eigenvalues and the lower bound eigenvalues $\underline{\lambda}_i, i = 1, 2, \ldots, n$, can be obtained by

$$\overline{\lambda}_i = \lambda_{i\max} = \max_{1\leq s\leq 2^{n\times n}}\{\lambda_i(\hat{A}_s)\}, \quad \underline{\lambda}_i = \lambda_{i\min} = \min_{1\leq s\leq 2^{n\times n}}\{\lambda_i(\hat{A}_s)\}, \quad i = 1, 2, \ldots, n, \tag{29}$$

where the eigenvalues $\lambda_{is}, i = 1, 2, \ldots, n, s = 1, 2, \ldots, 2^{n\times n}$ satisfy the following standard eigenvalue problems:

$$\hat{A}_s u_{is} = \lambda_{is}u_{is}, \ \ s = 1, 2, \ldots, 2^{n\times n}, \ \ i = 1, 2, \ldots, n, \tag{30}$$

*where $\hat{A}_s = (\hat{a}^s_{ij})$, and u_{is} is the eigenvector associated with the i*th *eigenvalue λ_{is}.*

5. Positive semi-definite solution theorem

To overcome the difficulty in determining the invariance properties of the eigenvectors' components of the interval matrix, and decrease the computational efforts, we presented the positive semi-definite solution theorem for the standard interval eigenvalue problem. This theorem is.

Positive semi-definite solution theorem. *If* $A^I = [\underline{A}, \overline{A}] = [A^c - \Delta A, A^c + \Delta A]$ *is a real symmetric interval matrix, and the deviation amplitude* k_{ij} *is a real positive semi-definite matrix, then the eigenvalues* $\lambda_i, i = 1, 2, \ldots, n$, *of* $A \in A^I$ *range over the intervals*

$$\lambda_i^I = [\underline{\lambda}_i, \overline{\lambda}_i], \quad i = 1, 2, \ldots, n, \tag{31}$$

where the lower bound eigenvalues $\underline{\lambda}_i, i = 1, 2, \ldots, n$, *satisfy*

$$\underline{A}\underline{u}_i = \underline{\lambda}_i \underline{u}_i \text{ or } (A^c - \Delta A)\underline{u}_i = \underline{\lambda}_i \underline{u}_i, \quad i = 1, 2, \ldots, n \tag{32}$$

in which $\underline{u}_i$ *is the eigenvector associated with the eigenvalue* $\underline{\lambda}_i$, *and the upper bound eigenvalues* $\overline{\lambda}_i, i = 1, 2, \ldots, n$, *satisfy*

$$\overline{A}\overline{u}_i = \overline{\lambda}_i \overline{u}_i \text{ or } (A^c + \Delta A)\overline{u}_i = \overline{\lambda}_i \overline{u}_i, \quad i = 1, 2, \ldots, n \tag{33}$$

in which $\overline{u}_i$ *is the eigenvector associated with the eigenvalue* $\overline{\lambda}_i$.

Proof. For the Courant and Fisher maxi–min theorem or the mini–max theorem of the matrix A

$$\lambda_i = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{u^T A u}{u^T u} \right\}, \quad i = 1, 2, \ldots, n \tag{34}$$

under the matrix constraint condition (4a), let us consider the extremum value problem as follows:

$$\lambda_{i\text{ext}} = \underset{A \in A^I}{\text{extremum}} \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{u^T A u}{u^T u} \right\} = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \underset{A \in A^I}{\text{extremum}} \left\{ \frac{u^T A u}{u^T u} \right\} \right\}, \quad i = 1, 2, \ldots, n. \tag{35}$$

Obviously, the eigenvalue λ_i is considered a function of the elements $a_{ij}, i, j = 1, 2, \ldots, n$, of the matrix $A = (a_{ij})$. Then, in terms of the natural interval extension [1,2], from Eq. (34) we can obtain

$$\lambda_i^I = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{u^T A^I u}{u^T u} \right\}, \quad i = 1, 2, \ldots, n. \tag{36}$$

It is assumed that the deviation amplitude matrix $\Delta A = (\overline{A} - \underline{A})/2$ is positive semi-definite. Then, for $u \in \Phi_i$ and $\overline{A} - \underline{A} = 2\Delta A$, we have

$$u^T(\overline{A} - \underline{A})u = 2u^T \Delta A u \geq 0 \tag{37}$$

which implies

$$u^T \overline{A} u \geq u^T \underline{A} u. \tag{38}$$

Considering $u^T u > 0$, by means of inequality (38) and the interval multiplication operation, Eq. (36) can be written as

$$\lambda_i^I = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \left[\frac{u^T \underline{A} u}{u^T u}, \frac{u^T \overline{A} u}{u^T u} \right] \right\}, \quad i = 1, 2, \ldots, n. \tag{39}$$

Further, bearing in mind that

$$\underline{A} = \overline{A} - 2\Delta A \tag{40}$$

we can get

$$\min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{u^{\mathrm{T}} \underline{A} u}{u^{\mathrm{T}} u} \right\} = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{u^{\mathrm{T}} (\overline{A} - 2\Delta A) u}{u^{\mathrm{T}} u} \right\} = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{u^{\mathrm{T}} \overline{A} u - 2u^{\mathrm{T}} \Delta A u}{u^{\mathrm{T}} u} \right\}$$

$$= \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{u^{\mathrm{T}} \overline{A} u}{u^{\mathrm{T}} u} \right\} - \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ 2 \frac{u^{\mathrm{T}} \Delta A u}{u^{\mathrm{T}} u} \right\} \leq \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{u^{\mathrm{T}} \overline{A} u}{u^{\mathrm{T}} u} \right\}. \quad (41)$$

Hence, from Eq. (39), we obtain

$$\lambda_i^I = [\underline{\lambda}_i, \overline{\lambda}_i] = \left[\min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{u^{\mathrm{T}} \underline{A} u}{u^{\mathrm{T}} u} \right\}, \ \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{u^{\mathrm{T}} \overline{A} u}{u^{\mathrm{T}} u} \right\} \right], \quad i = 1, 2, \ldots, n. \quad (42)$$

According to the necessary and sufficient conditions of equality of interval variables, we have that

$$\overline{\lambda}_i = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{u^{\mathrm{T}} \overline{A} u}{u^{\mathrm{T}} u} \right\}, \quad i = 1, 2, \ldots, n \quad (43)$$

and

$$\underline{\lambda}_i = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{u^{\mathrm{T}} \underline{A} u}{u^{\mathrm{T}} u} \right\}, \quad i = 1, 2, \ldots, n. \quad (44)$$

Since the stationary condition of the Rayleigh quotient is equivalent to the algebraic eigenvalue problem, the eigenvalue problem corresponding to the upper bound of Eq. (43) is

$$\overline{A} \overline{u}_i = \overline{\lambda}_i \overline{u}_i, \quad i = 1, 2, \ldots, n, \quad (45)$$

where $\overline{u}_i$ is the eigenvector associated with the eigenvalue $\overline{\lambda}_i$. Similarly, the eigenvalue problem corresponding to the lower bound of Eq. (44) is

$$\underline{A} \underline{u}_i = \underline{\lambda}_i \underline{u}_i, \quad i = 1, 2, \ldots, n, \quad (46)$$

where $\underline{u}_i$ is the eigenvector associated with the eigenvalue $\underline{\lambda}_i$.

Thus, we complete the proof of the positive semi-definite solution theorem.

6. The parameter decomposition solution theorem

As we know, in most engineering problems, the real symmetric matrix A may be thought of as a function of the structural parameter b=$(b_i)_m$, that is

$$A = A(b). \quad (47)$$

Let us consider the algebraic eigenvalue problem (3) subject to the parameter constraint condition

$$\underline{b} \leq b \leq \overline{b} \quad \text{or} \quad \underline{b}_i \leq b_i \leq \overline{b}_i, \quad i = 1, 2, \ldots, m. \quad (48)$$

By means of the structural parameter vector $b=(b_i)_m$, the real symmetric matrix A can be expressed in following form:

$$A(b) = \sum_{i=1}^{m} b_i A_i = b_1 A_1 + b_2 A_2 + \cdots + b_m A_m, \tag{49}$$

where A_i is the symmetric matrix associated with the structural parameter b_i.

In a practical engineering context, it is simple for the kind of the decomposition. For example, in structural finite element analysis, A_i may be taken as the element stiffness matrices corresponding to the structural parameter b_i. In the substructure method, A_i may be taken as the substructure matrices corresponding to the structural parameter b_i.

Clearly, the elements $a_{ij}, i,j = 1,2,\ldots,n$, of the real symmetric matrix A are also functions of the structural parameters $b=(b_i)_m$. Then in terms of the natural interval extension, from Eq. (48), we can obtain

$$A^I = [\underline{A}, \overline{A}] = \sum_{i=1}^{m} b_i^I A_i = b_1^I A_1 + b_2^I A_2 + \cdots + b_m^I A_m, \tag{50}$$

where $b_i^I = [\underline{b}_i, \overline{b}_i], i = 1,2,\ldots,m$, are the interval parameters. In terms of interval operations and the definition for equality of intervals, we have

$$\underline{A} = (\underline{a}_{ij}), \quad \overline{A} = (\overline{a}_{ij}) \tag{51}$$

where

$$\underline{a}_{ij} = \min\{\underline{b}_i a_{ij}, \overline{b}_i a_{ij}\}, \quad \overline{a}_{ij} = \max\{\underline{b}_i a_{ij}, \overline{b}_i a_{ij}\}. \tag{52}$$

From the process of calculating, we can deduce that if the interval parameters $b_i^I = [\underline{b}_i, \overline{b}_i], i = 1,2,\ldots,m$, are precise, the interval stiffness matrix $A^I = [\underline{A}, \overline{A}]$ is also precise.

In engineering practice, there exists some cases: A and $A_i, i = 1,2,\ldots,m$ may be all positive definite matrices, but $\overline{A}$ and $\underline{A}$ are not necessarily positive definite matrices, and if the width $w(b^I) = (\overline{b} - \underline{b})$ of the interval parameter $b^I = [\underline{b}, \overline{b}]$ is large enough, $\underline{A}$ may be negative definited. This is one reason why the width of the eigenvalue by the solution to the interval eigenvalue problem is much larger.

In order to obtain the sharp bounds on the eigenvalues, we shall combine interval analysis with real analysis. Let us concentrate on the following expression of the real symmetric matrices:

$$\underline{\underline{A}} = \sum_{i=1}^{m} \underline{b}_i A_i, \quad \overline{\overline{A}} = \sum_{i=1}^{m} \overline{b}_i A_i. \tag{53}$$

In Eq. (53), the following relations do not generally hold:

$$\underline{\underline{A}} \leq \overline{\overline{A}}. \tag{54}$$

Obviously, $\underline{\underline{A}}$ and $\overline{\overline{A}}$ are real symmetric matrices.

Under the constraint condition (48), let us consider the extremum value problem of the mini–max eigenvalues (35).

Substituting Eq. (49) into Eq. (35), we arrive at

$$\lambda_i = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \underset{b \in b^I}{\text{extremum}} \left\{ \frac{u^{\mathrm{T}} \left(\sum_{i=1}^{m} b_i A_i \right) u}{u^{\mathrm{T}} u} \right\} \right\} = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \underset{b \in b^I}{\text{extremum}} \left\{ \frac{\sum_{i=1}^{m} b_i \left(u^{\mathrm{T}} A_i u \right)}{u^{\mathrm{T}} u} \right\} \right\},$$
$$i = 1, 2, \ldots, n. \tag{55}$$

Since $b_i, i = 1, 2, \ldots, m$, are the interval parameters, by means of the interval natural extension, from Eq. (55), we obtain

$$\lambda_i^I = [\underline{\lambda}_i, \overline{\lambda}_i] = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{\sum_{i=1}^{m} b_i^I (u^{\mathrm{T}} A_i u)}{u^{\mathrm{T}} u} \right\} = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{\sum_{i=1}^{m} [\underline{b}_i, \overline{b}_i] (u^{\mathrm{T}} A_i u)}{u^{\mathrm{T}} u} \right\},$$
$$i = 1, 2, \ldots, n. \tag{56}$$

Since $u^{\mathrm{T}} A_i u \geq 0$, by interval multiplications, from Eq. (56), we get

$$\lambda_i^I = [\underline{\lambda}_i, \overline{\lambda}_i] = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{\sum_{i=1}^{m} [\underline{b}_i (u^{\mathrm{T}} A_i u), \overline{b}_i (u^{\mathrm{T}} A_i u)]}{u^{\mathrm{T}} u} \right\}, \quad i = 1, 2, \ldots, n. \tag{57}$$

From the numerator of Eq. (57), using interval additions, we reach

$$\begin{aligned} \lambda_i^I = [\underline{\lambda}_i, \overline{\lambda}_i] &= \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{\left[\sum_{i=1}^{m} \underline{b}_i (u^{\mathrm{T}} A_i u), \sum_{i=1}^{m} \overline{b}_i (u^{\mathrm{T}} A_i u) \right]}{u^{\mathrm{T}} u} \right\} \\ &= \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{\left[u^T \left(\sum_{i=1}^{m} \underline{b}_i A_i \right) u, u^{\mathrm{T}} \left(\sum_{i=1}^{m} \overline{b}_i A_i \right) u \right]}{u^{\mathrm{T}} u} \right\}, \quad i = 1, 2, \ldots, n. \end{aligned} \tag{58}$$

Substituting Eqs. (53) into Eq (58), we have

$$\lambda_i^I = [\underline{\lambda}_i, \overline{\lambda}_i] = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \frac{[u^{\mathrm{T}} \underline{\underline{A}} u, u^{\mathrm{T}} \overline{\overline{A}} u]}{[u^{\mathrm{T}} u, u^{\mathrm{T}} u]} \right\}, \quad i = 1, 2, \ldots, n. \tag{59}$$

Since $u^{\mathrm{T}} u > 0$, by the interval division, from Eq. (59), we obtain

$$\lambda_i^I = [\underline{\lambda}_i, \overline{\lambda}_i] = \min_{\Phi_i \subset R^n} \max_{\substack{u \in \Phi_i \\ u \neq 0}} \left\{ \left[\frac{u^{\mathrm{T}} \underline{\underline{A}} u}{u^{\mathrm{T}} u}, \frac{u^{\mathrm{T}} \overline{\overline{A}} u}{u^{\mathrm{T}} u} \right] \right\}, \quad i = 1, 2, \ldots, n. \tag{60}$$

By the meaning of the interval number, from the interval number $\left[u^{\mathrm{T}}\underline{\underline{A}}u\Big/u^{\mathrm{T}}u,\ u^{\mathrm{T}}\overline{\overline{A}}u\Big/u^{\mathrm{T}}u\right]$, we arrive at

$$\frac{u^{\mathrm{T}}\underline{\underline{A}}u}{u^{\mathrm{T}}u} \le \frac{u^{\mathrm{T}}\overline{\overline{A}}u}{u^{\mathrm{T}}u}. \tag{61}$$

Thus, we can deduce that the following expression holds:

$$\min_{\Phi_i\subset R^n}\ \max_{\substack{u\in\Phi_i\\ u\neq 0}}\left\{\frac{u^{\mathrm{T}}\underline{\underline{A}}u}{u^{\mathrm{T}}u}\right\} \le \min_{\Phi_i\subset R^n}\ \max_{\substack{u\in\Phi_i\\ u\neq 0}}\left\{\frac{u^{\mathrm{T}}\overline{\overline{A}}u}{u^{\mathrm{T}}u}\right\}, \quad i=1,2,\ldots,n. \tag{62}$$

Hence, we obtain

$$\lambda_i^I=[\underline{\lambda}_i,\overline{\lambda}_i]=\left[\min_{\Phi_i\subset R^n}\ \max_{\substack{u\in\Phi_i\\ u\neq 0}}\left\{\frac{u^{\mathrm{T}}\underline{\underline{A}}u}{u^{\mathrm{T}}u}\right\},\ \min_{\Phi_i\subset R^n}\ \max_{\substack{u\in\Phi_i\\ u\neq 0}}\left\{\frac{u^{\mathrm{T}}\overline{\overline{A}}u}{u^{\mathrm{T}}u}\right\}\right], \quad i=1,2,\ldots,n. \tag{63}$$

According to the necessary and sufficient conditions of the equality of interval variables [13], we have

$$\underline{\lambda}_i=\min_{\Phi_i\subset R^n}\ \max_{\substack{u\in\Phi_i\\ u\neq 0}}\left\{\frac{u^{\mathrm{T}}\underline{\underline{A}}u}{u^{\mathrm{T}}u}\right\}, \quad i=1,2,\ldots,n \tag{64}$$

and

$$\overline{\lambda}_i=\min_{\Phi_i\subset R^n}\ \max_{\substack{u\in\Phi_i\\ u\neq 0}}\left\{\frac{u^{\mathrm{T}}\overline{\overline{A}}u}{u^{\mathrm{T}}u}\right\}, \quad i=1,2,\ldots,n. \tag{65}$$

The stationary condition of Rayleigh's quotient is equivalent to the algebraic eigenvalue problem. Thus, the eigenvalue problem corresponding to the lower bound of Eq. (64) reads

$$\underline{\underline{A}}\underline{u}_i=\underline{\lambda}_i\underline{u}_i, \quad i=1,2,\ldots,n, \tag{66}$$

where $\underline{u}_i$ is the eigenvector associated with $\underline{\lambda}_i$.

Similarly, the eigenvalue problem corresponding to the upper bound of Eq. (65) is given by

$$\overline{\overline{A}}\overline{u}_i=\overline{\lambda}_i\overline{u}_i, \quad i=1,2,\ldots,n, \tag{67}$$

where $\overline{u}_i$ is the eigenvector associated with $\overline{\lambda}_i$.

Thus, we arrive at the following solution theorem:

The parameter decomposition solution theorem. *Let the eigenvalues $\lambda_i, i=1,2,\ldots,n$, be the function of parameters $b_i, i=1,2,\ldots,m$, i.e. $\lambda_i=\lambda_i(b_1,b_2,\ldots,b_m), i=1,2,n$. If the real symmetric matrix A can be decomposed as $A=\sum_{i=1}^m b_iA_i$, and the parameters $b_i, i=1,2,\ldots,m$, are interval parameters, i.e. $b_i^I=[\underline{b}_i,\overline{b}_i], i=1,2,\ldots,m$, then eigenvalues $\lambda_i, i=1,2,\ldots,n$, range over the interval*

$$\lambda^I=[\underline{\lambda},\overline{\lambda}]=(\lambda_i^I), \quad \lambda_i^I=[\underline{\lambda}_i,\overline{\lambda}_i], \quad i=1,2,\ldots,n, \tag{68}$$

where the lower bounds $\underline{\lambda}_i$ satisfy

$$\underline{\underline{A}}\underline{u}_i = \underline{\lambda}_i \underline{u}_i, \quad i = 1, 2, \ldots, n \tag{69}$$

in which $\underline{\underline{A}} = \sum_{i=1}^{m} \underline{b}_i A_i$, $\underline{u}_i$ is the eigenvector associated with $\underline{\lambda}_i$ and the upper bounds $\overline{\lambda}_i$ satisfy

$$\overline{\overline{A}}\overline{u}_i = \overline{\lambda}_i \overline{u}_i, \quad i = 1, 2, \ldots, n, \tag{70}$$

where $\overline{\overline{A}} = \sum_{i=1}^{m} \overline{b}_i A_i$, and $\overline{u}_i$ is the eigenvector associated with $\overline{\lambda}_i$.

7. Numerical examples

A plate and a truss structure are analyzed to illustrate the validity of the proposed vertex solution theorem, the positive semi-definite solution theorem and the parameter decomposition solution theorem in this paper. Numerical examples consist of a flat square plate and an eight-bar truss. In each problem, some structural parameters are taken as uncertain variables and the others are thought of as deterministic variables. Because the standard interval eigenvalue problem is discussed, our object is to calculate the interval eigenvalues or the upper and lower bounds on eigenvalues of the dynamic matrix $A = M^{-1}K$ of the structural system with uncertain-but-bounded parameters, and to compare the presented vertex solution theorem, the positive semi-definite solution theorem and the parameter decomposition solution theorem with Deif's solution theorem in computational aspects and accuracies.

In the two numerical examples, no matter how the uncertain factor β changes, the upper bounds predicted by the vertex solution theorem are equal to the upper bounds determined by the Deif's solution theorem for the eigenvalues of structures, and they are denoted by $\overline{\lambda}_i$. Likewise, the lower bounds produced by the vertex solution theorem are equal to the lower bounds calculated by the Deif's solution theorem for the eigenvalues of structures, and they are denoted by $\underline{\lambda}_i$. The upper and lower bounds of the eigenvalues obtained by the positive semi-definite solution theorem are, respectively, represented by $\overline{\mu}_i$ and $\underline{\mu}_i$. The upper and lower bounds of the eigenvalues obtained by the parameter decomposition theorem are, respectively, expressed by $\overline{\gamma}_i$ and $\underline{\gamma}_i$.

7.1. Example I. A flat square plate

The first numerical example deals with the eigenvalue analysis of the system dynamic matrix of a flat square plate with uncertain-but-bounded parameters as shown in Fig. 1. In this example, the Young's modulus E, the Poisson ratio v and the density ρ of the plate are considered uncertain-but-bounded parameters, and they are expressed by, respectively: $E^I = [E^c - 2\beta E^c, E^c + 2\beta E^c]$, $v^I = [v^c - \beta v^c, v^c + \beta v^c]$ and $\rho^I = [\rho^c - 2\beta\rho^c, \rho^c + 2\beta\rho^c]$, where $E^c = 2.1 \times 10^{11}$ N/m^2, $v^c = 0.3$, $\rho^c = 7800.0$ kg/m^3, and β is the uncertain factor. The other structural parameters are taken as deterministic values, and they are: the width of the plate $L = 0.3$ m; the thickness of the plate where $t = 1$ mm. The interval eigenvalues or the upper and lower bounds on the eigenvalues of the dynamic matrix of the flat square plate, which are calculated by the Deif's solution theorem, the proposed vertex solution theorem and the positive semi-definite solution theorem, are plotted in Fig. 3. Only the first eight order eigenvalues are given.

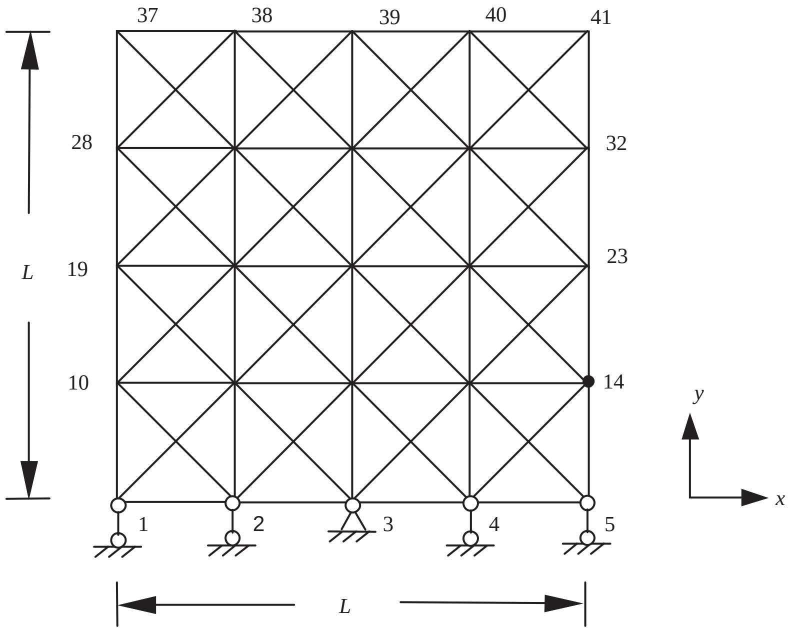

Fig. 1. A flat square plate divided by triangle constant–strain elements.

From Fig. 3, it can be seen that for the any order eigenvalue of the dynamic matrix of the flat square plate, the following inequality holds

$$\underline{\lambda}_i \le \underline{\mu}_i \le \overline{\mu}_i \le \overline{\lambda}_i, \quad i = 1, 2, \ldots, 8.$$

7.2. Example II. Eight-bar truss

The second numerical example deals with the interval eigenvalues of the dynamic matrix of the eight-bar truss with uncertain-but-bounded parameters, as shown in Fig. 2. In this numerical example, the uncertain-but-bounded parameter are taken as the Young's moduli of the bars of the eight-bar truss, they are: $E_i^I = [E^c - \beta E^c, E^c + \beta E^c], i = 1, 2, \ldots, 8$, where β is the uncertain factor and $E^c = 2.1 \times 10^{11}$ N/m^2. The other structural parameters are considered as deterministic variables, they are the cross-sectional areas of the other bars: $A_i = 2.0 \times 10^{-3}$ m$^2, i = 1, 2, 3, 4, 6$, $A_i = 1.0 \times 10^{-3}$ m$^2, i = 5, 7, 8$ and the mass density: $\rho = 7800.0$ kg/m^3 (see also Fig. 3).

The interval eigenvalues or the upper and lower bounds on the eigenvalues of the dynamic matrix of the eight-bar truss, which are calculated by the Deif's solution theorem and the proposed vertex solution theorem, the positive semi-definite solution theorem and the parameter decomposition solution theorem, are shown in Fig. 4.

From Fig. 4, it can be seen that for the i=1,2,4,5,6,7 order eigenvalue of the dynamic matrix of the eight-bar truss, the following inequalities hold:

$$\underline{\lambda}_i \le \underline{\mu}_i \le \underline{\gamma}_i \le \overline{\gamma}_i \le \overline{\mu}_i \le \overline{\lambda}_i.$$

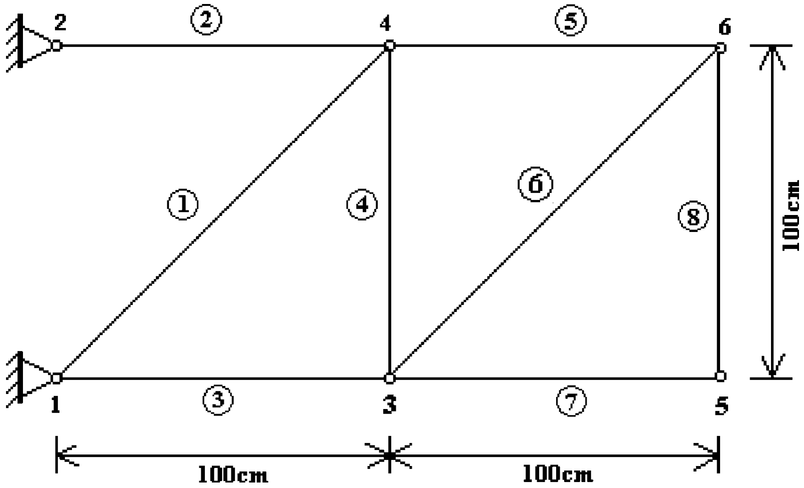

Fig. 2. An eight bar truss.

For the $i=3$ order eigenvalue, the following inequality holds:

$$\underline{\lambda}_3 = \underline{\mu}_3 \leq \underline{\gamma}_3 \leq \overline{\gamma}_3 \leq \overline{\mu}_3 = \overline{\lambda}_3.$$

However, for the $i=8$ order eigenvalue, the following inequality holds:

$$\underline{\lambda}_8 = \underline{\gamma}_8 \leq \underline{\mu}_8 \leq \overline{\mu}_8 \leq \overline{\gamma}_8 = \overline{\lambda}_8.$$

In the two cases, from Figs. 3 and 4, we can see that as one might expect the widths of the upper and lower bounds on the eigenvalues of the dynamic matrices of the structural systems show growth with increased uncertainty of the uncertain factor or the structural parameters. It is observed that the widths of the upper and lower bounds, which are calculated by the parameter decomposition theorem, on the eigenvalues of the dynamic matrices of the structural systems get larger and larger with the order of the eigenvalues increasing. However, the widths of the upper and lower bounds, which are obtained by the positive semi-definite theorem, on the eigenvalues of the stiffness matrices of the structural systems get smaller and smaller with the order of the eigenvalues increasing.

8. Conclusions

In this paper, uncertainties in structural parameters are considered through the so-called non-probabilistic interval formulations. Uncertain structural parameters represented by interval numbers, the vertex solution theorem, the positive semi-definite solution and the parameter decomposition solution theorem have been presented to compute the upper and lower bounds on the eigenvalues of the standard eigenvalue problem of structures with uncertain-but-bounded parameters. The respective performances of the presented theorems have been discussed. We presented the vertex solution theorem a unconditional, and the positive semi-definite solution theorem and the parameter decomposition solution theorem have less limitary condition

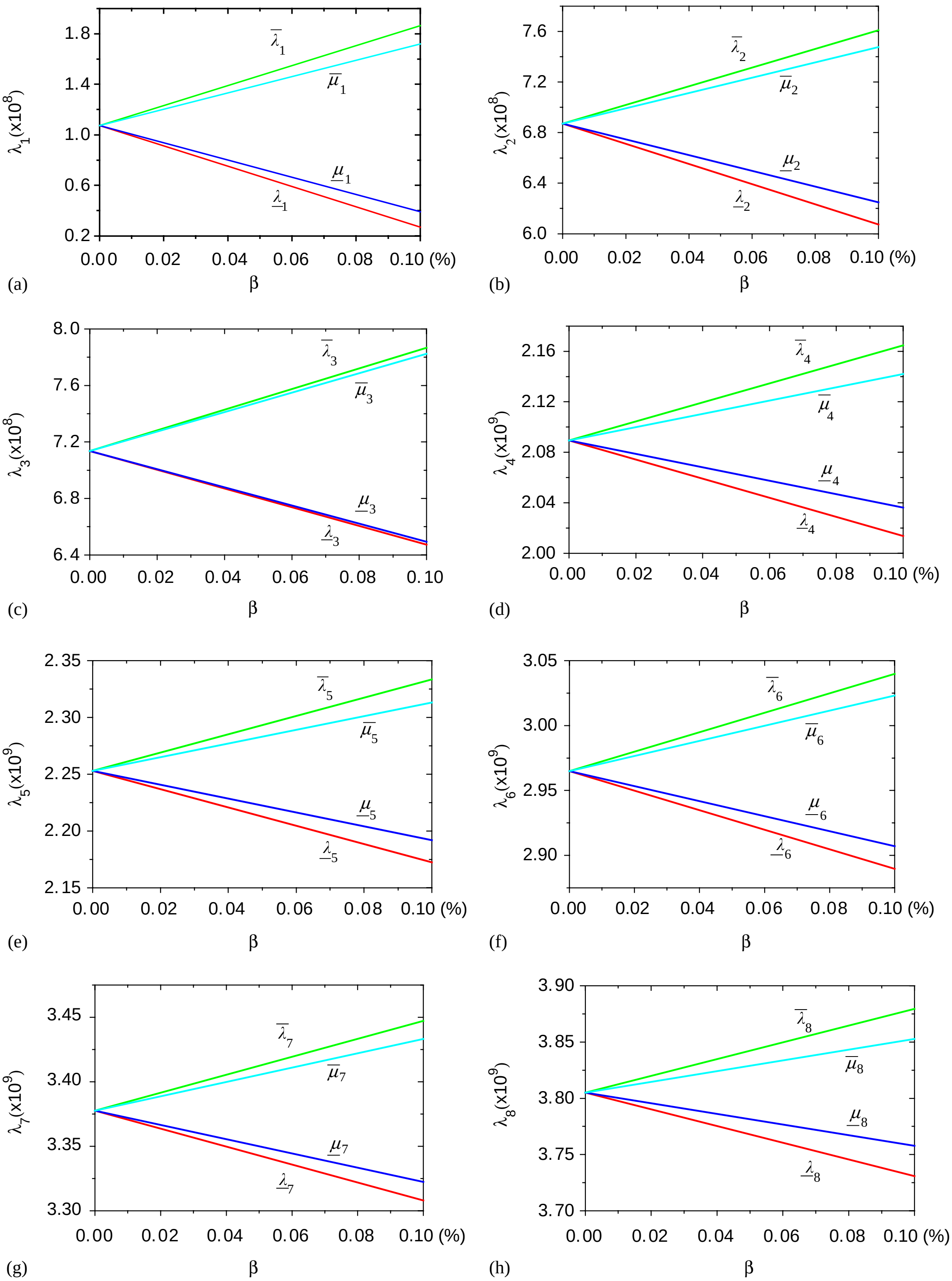

Fig. 3. Interval eignvalues of a flat square plate with the factor β.

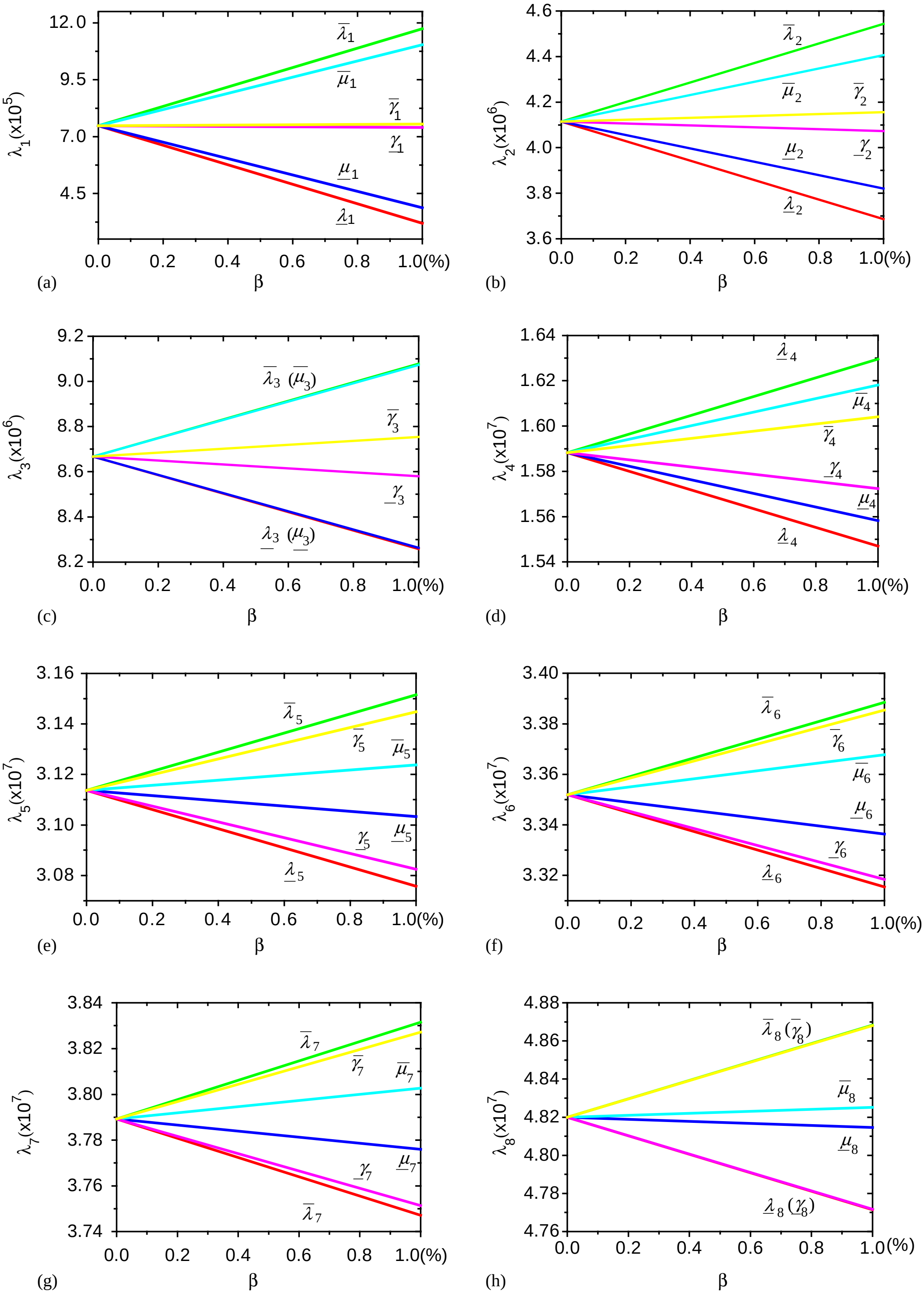

Fig. 4. Interval eigenvalues of an eight-bar truss with the factor β.

compared with Deif's solution theorem. By comparing with Deif's solution theorem, numerical examples are presented to illustrate the effectiveness of the vertex solution theorem, the positive semi-definite solution and the parameter decomposition solution theorem for computing the upper and lower bounds on the eigenvalues of the standard eigenvalue problem of structures with uncertain-but-bounded parameters. In order to treat the eigenvalue problem entirely, the problem of the eigenvector should also be solved. The effects of the uncertainties or imprecision of the structure's parameters on the eigenvalues of structures need further study.

Acknowledgment

The work of Zhiping, Qiu was part by supported by the National Natural Science Foundation of the P R China and Institute of Engineering Physics of the P R China and the Aeronautical Science Foundation of the P R China.

References

[1] R.E. Moore, *Methods and Applications of Interval Analysis*, Prentice-Hall, London, 1979.
[2] G. Alefeld, J. Herzberger, *Introductions to Interval Computations*, Academic Press, New York, 1983.
[3] J. Rohn, Eigenvalues of a symmetric interval matrix, Freiburger Intervall-Berichte, B7/1d, 1987, pp.67–72.
[4] C. Hallot, A. Bartlett, On the eigenvalues of interval matrices, Technical Report, Department of Electric Engineering and Computer Engineering, University of Massachusetts, Amherst, MA, USA, 1987.
[5] D. Hudak, Ein Eigenwertproblem für Intervall-Matrizen, *Journal of Applied Mathematics and Mechanics* (*ZAMM*) 64 (1984) 503–505.
[6] A. Deif, *Advanced Matrix Theory for Scientists and Engineers*, Abacus Press, Tunbridge Wells, UK, 1991, pp. 262–281.
[7] Z. P. Qiu, S. H. Chen, J. X. Na, The Rayleigh quotient method for computing eigenvalue bounds of vibrational systems with interval parameters, *Acta Mechanics Solid Sinica* 6(3) (1993) 309–318 (English edition).
[8] Z.P. Qiu, S.H. Chen, I. Elishakoff, Natural frequencies of structures with uncertain-but-non-random parameters, *Journal of Optimization Theory and Applications* 86 (3) (1995) 669–683.
[9] Z.P. Qiu, S.H. Chen, I. Elishakoff, Bounds of eigenvalus for structures with an interval description of uncertain-but-non-random parameters, *Chaoos, Solitons & Fractals* 7 (3) (1996) 425–434.
[10] H. C. Hu, *The Theory of Natural Vibration of MDOF Structures*, Science Press of China, Beijing, 1987, pp.1–80, (in Chinese).
[11] P. Lancaster, *Theory of Matrices*, Academic Press, New York, 1985, pp. 289–380.